Worked Example: Distinct Real Roots

Problem. Find the general solution to

$$\dot{\mathbf{u}} = A\mathbf{u}, \quad \text{where} \quad A = \begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix}.$$

Find the solution with initial conditions $\mathbf{u}(0) = (1,0)^T$. *Throughout, comments are given in italics.*

Solution.
Step 0. Write down $A - \lambda I$
Even if you find the characteristic equation of A using its trace and determinant, you will need this later, for finding eigenvectors. Most students find it useful to write it down clearly at the start of the question.

$$A - \lambda I = \begin{pmatrix} -2-\lambda & 1 \\ -4 & 3-\lambda \end{pmatrix}.$$

Step 1. Find the characteristic equation of A.
We use the method involving the trace and determinant of A.

$$\begin{aligned} \operatorname{tr}(A) &= -2+3 = 1 \\ \det(A) &= -2 \times 3 - 1 \times (-4) = -6+4 = -2 \end{aligned}$$

Thus $p_A(\lambda) = \det(A - \lambda I) = \lambda^2 - \lambda - 2$.

Step 2. Find the eigenvalues of A.
These are the roots of the characteristic equation. we complete the square. (We could also have used the quadratic formula.)

$$p_A(\lambda) = (\lambda - 1/2)^2 - 9/4.$$

The roots are $1/2 \pm 3/2$, so $\lambda_1 = -1$ and $\lambda_2 = 2$.

Step 3. Find associated eigenvectors.
3a. Eigenvector for λ_1. This is vector $\mathbf{a} = (a_1, a_2)^T$ that must satisfy

$$\begin{aligned} (A+I)\mathbf{a} = 0 \quad &\Leftrightarrow \quad \begin{pmatrix} -2+1 & 1 \\ -4 & 3+1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \quad \begin{pmatrix} -1 & 1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \quad \left\{ \begin{array}{rcrcl} -a_1 & + & a_2 & = & 0 \\ -4a_1 & + & a_2 & = & 0 \end{array} \right. \end{aligned}$$

Check: one equation is a multiple of the other, as should be the case. This is a good sign. Setting $a_1 = 1$ gives $a_2 = 1$; thus one eigenvector for λ_1 is $(1,1)^T$.

3b. Eigenvector for λ_2. This is a vector $(a_1, a_2)^T$ that must satisfy:

$$(A - 2I)\mathbf{a} = 0 \Leftrightarrow \begin{pmatrix} -2-2 & 1 \\ -4 & 3-2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{array}{rcrcl} -4a_1 & + & a_2 & = & 0 \\ -4a_1 & + & a_2 & = & 0 \end{array}$$

Check: one equation is a (trivial) multiple of the other.
Setting $a_1 = 1$ gives $a_2 = 4$. Thus, one eigenvector for λ_2 is $(1,4)^T$.

Step 4. Normal modes and general solution

The normal modes are $e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $e^{2t}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$.
and the general solution is:

$$\mathbf{u}(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

Step 5. Solution matching IC.

We solve for c_1 and c_2 using our initial condition. From our expression for the general solution, $\mathbf{u}(0) = c_1(1,1)^T + c_2(1,4)^T = (c_1 + c_2, c_1 + 4c_2)^T$. Thus the initial condition $\mathbf{u}(0) = (1,0)^T$ gives:

$$\begin{array}{rcrcl} c_1 & + & c_2 & = & 1 \\ c_1 & + & 4c_2 & = & 0 \end{array} \quad \Leftrightarrow \quad c_2 = -1/3,\ c_1 = 4/3$$

The solution we were asked for is:

$$\mathbf{u}(t) = \frac{4}{3}e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{3}e^{2t}\begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

18.03SC Differential Equations
Fall 2011